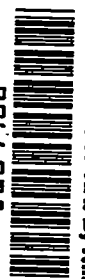


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# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3318

ON THE SMALL-DISTURBANCE ITERATION METHOD FOR THE FLOW  
OF A COMPRESSIBLE FLUID WITH APPLICATION  
TO A PARABOLIC CYLINDER

By Carl Kaplan

Langley Aeronautical Laboratory  
Langley Field, Va.



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## SUMMARY

The assumptions of the Prandtl-Busemann small-disturbance method, together with the requirements of continuity and irrotationality, lead to a recursive system of first-order partial-differential equations. The first three sets of equations of this iterative procedure are rewritten in complex-vector form and readily integrated for their particular integrals. The results of the general analysis are then applied to the case of subsonic flow past a parabolic cylinder. This calculation shows that the curtailed small-disturbance solution, without the restraining influence of a control parameter, is unsuitable for the description of subsonic flow past the parabolic cylinder. When, however, the small-disturbance solution is developed in powers of the undisturbed stream Mach number  $M_\infty$  as a control parameter and compared with the solution obtained by means of the Janzen-Rayleigh or  $M_\infty^2$ -expansion method, the two results are identical. This agreement shows that the Prandtl-Busemann and Janzen-Rayleigh developments are but two different arrangements of the actual solution. Finally, the small-disturbance solution for the parabolic cylinder is examined from the point of view of thin-airfoil theory. The series development of the fluid speed at the surface in powers of the ratio of the radius of curvature at the vertex and the abscissa measured from the vertex agrees with the results of second-order thin-airfoil theory. Also, a third-order thin-airfoil approximation is proposed.

## INTRODUCTION

The problem of the integration of the equations of compressible flow past a prescribed solid boundary has been treated most often by two approximation methods. The first one, initiated by Janzen and Rayleigh, proceeds from the incompressible complex potential and develops the compressibility effects in a series of powers of the undisturbed stream Mach number. It is restricted to the subsonic range because the differential equations of the process are always of the elliptic type. This method, moreover, is

suited particularly for thick bodies and, hence, for relatively small critical stream Mach numbers. The second one, the Prandtl-Busemann small-disturbance method, proceeds from the undisturbed stream and determines the disturbance effects by an expansion in series according to a geometric parameter characteristic of the body shape. This method is most suitable for slender bodies for which the critical stream Mach numbers are close to unity. Very little is known about the limit of convergence of the power series employed in these two methods. In a recent paper, however, on high subsonic flow past a sinusoidal wall, a plausible argument was presented which indicated that the limit of convergence is coincident with the attainment of local sonic velocity (ref. 1).

The present paper contains a brief account of Imai's elegant version of the Prandtl-Busemann small-disturbance method (ref. 2). The original Prandtl-Busemann method is based on the following assumption: If  $\epsilon$  is a parameter that characterizes the departure of the profile shape from a straight-line segment (for example, thickness, camber, or angle of attack) at zero incidence, the velocity potential  $\phi$  and the stream function  $\psi$  can be represented by series in powers of  $\epsilon$ , the coefficients of which are functions of the flow-plane coordinates  $x$  and  $y$ . Thus, in non-dimensional form:

$$\left. \begin{aligned} \phi &= -x + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots \\ \psi &= -y + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \dots \end{aligned} \right\} \quad (1)$$

where the undisturbed stream is directed from right to left. On the basis of this assumption, there follows from the general second-order nonlinear compressible-flow equation for  $\phi$  or  $\psi$ , by means of a comparison of coefficients, a recursive system of second-order differential equations for the coefficients of the individual powers of  $\epsilon$ . The first one is a Laplace type of equation and the ones that follow are of the Poisson type, the right-hand sides of which are composed of previously determined functions.

Imai's version of the Prandtl-Busemann method proceeds from the set of first-order differential equations for  $\phi$  and  $\psi$  that results from the requirements of continuity and irrotationality. Thus,

$$\left. \begin{aligned} u &= \phi_x = \frac{\rho_\infty}{\rho} \psi_y \\ v &= \phi_y = -\frac{\rho_\infty}{\rho} \psi_x \end{aligned} \right\} \quad (2)$$

Then, by means of equations (1) and the following expression for  $\frac{\rho}{\rho_\infty}$ ,

$$\frac{\rho}{\rho_\infty} = \left[ 1 - \frac{\gamma - 1}{2} M_\infty^2 (q^2 - 1) \right]^{\frac{1}{\gamma-1}} \quad (3)$$

a comparison of the various powers of  $\epsilon$  yields a recursive system of first-order equations for  $\phi_1, \psi_1; \phi_2, \psi_2; \phi_3, \psi_3; \dots$  similar to equations (2). These pairs of equations can be expressed in complex-vector form and readily integrated for their particular integrals. The symbols used in the preceding equations are defined as follows:

- x,y    rectangular Cartesian coordinates in flow plane
- $\phi$     velocity potential
- $\psi$     stream function
- u,v    velocity components in direction of x- and y-axes, respectively
- $\rho$     density of fluid
- $\rho_\infty$     density of fluid in undisturbed stream
- q    fluid speed
- $M_\infty$     Mach number of undisturbed flow
- $\gamma$     ratio of specific heats at constant pressure and volume

The quantities  $x, y, \phi, \psi, u, v$ , and  $q$  are all nondimensional and the subscripts denote differentiation with respect to the designated variable.

Whereas in the Janzen-Rayleigh method the expansion of the complex potential in powers of  $M_\infty^2$  is always possible, in the Prandtl-Busemann method the expansion of  $\psi$  (or  $\phi$ ) in the form of equations (1) cannot be guaranteed a priori. Indeed, experience has shown that only in cases where the prescribed profile has no stagnation points so that the assumptions of the small-disturbance method are adhered to strictly can the expansions for  $\phi$  and  $\psi$ , as indicated in equations (1), be valid (refs. 3 and 4). Most profiles of aeronautical interest, however, have rounded leading edges and therefore possess stagnation points, in the neighborhood of which the assumption of small deviation from undisturbed flow is clearly violated. In such cases, sooner or later, terms of the form  $\epsilon^n \log \epsilon$  must appear (ref. 5). The procedure then is as follows: The velocity

potential and stream functions are still assumed in the form of equations (1), but with the parameter  $\epsilon$  now considered as a dummy symbol which serves only to regulate the course of the iteration process. When the stream function only is considered, the approximation functions  $\psi_n$  ( $n \geq 2$ ) are assumed to satisfy the boundary condition that they vanish on the contour, are regular in the region of flow outside the boundary, and their derivatives vanish to a sufficient degree at infinity. The first pair of terms in the expansion usually represents the Prandtl-Glauert linearized approximation. It is further assumed that, except for a small region in the neighborhood of stagnation points, the function  $\psi_n$  and its derivatives are small compared with  $\psi_m$  and its derivatives for all  $m < n$ . The question whether this assumption is satisfied can be answered only after the  $\psi_n$  have been calculated.

As borne out affirmatively in the only two cases thus far calculated, namely, the elliptic cylinder and the circular-arc profile (refs. 5 and 6), it seems reasonable to conjecture that, if it were possible to calculate  $\psi_n$  to any order and if each term were developed in powers of  $M_\infty^2$ , the formal arrangement in series of these powers would yield precisely the Janzen-Rayleigh solution for the same profile. The example chosen to illustrate the general analysis of the present paper and to verify the preceding conjecture is the flow at zero incidence past a parabolic cylinder. This profile is especially amenable to treatment by means of both the small-disturbance method and the Janzen-Rayleigh method. Moreover, because the parabolic cylinder represents a magnified picture of a stagnation region, it is particularly well suited for a critical examination of the utility of the small-disturbance method for the calculation of the flow in the neighborhood of stagnation points. It does not follow, even though the preceding conjecture is verified, that the small-disturbance method provides a suitable approximation in the neighborhood of a stagnation point.

#### ANALYSIS

The development of the nondimensional fluid speed  $q$  is obtained with the aid of equations (1) and (2). Thus,

$$q^2 - 1 = -2\epsilon\phi_{1x} + \epsilon^2(\phi_{1x}^2 + \phi_{1y}^2 - 2\phi_{2x}) + \\ 2\epsilon^3(\phi_{1x}\phi_{2x} + \phi_{1y}\phi_{2y} - \phi_{3x}) + \dots \quad (4)$$

The corresponding development for  $\frac{\rho_\infty}{\rho}$  is obtained by means of equations (3) and (4); that is,

$$\begin{aligned} \frac{\rho_\infty}{\rho} = & 1 - \epsilon M_\infty^2 \phi_{1x} + \frac{1}{2} \epsilon^2 M_\infty^2 \left[ (1 + \gamma M_\infty^2) \phi_{1x}^2 + \phi_{1y}^2 - 2\phi_{2x} \right] + \\ & \epsilon^3 M_\infty^2 \left[ \phi_{1x} \phi_{2x} + \phi_{1y} \phi_{2y} - \phi_{3x} + \frac{1}{2} \gamma M_\infty^2 (-\phi_{1x} \phi_{1y}^2 + 2\phi_{1x} \phi_{2x}) - \right. \\ & \left. \frac{1}{2} \gamma M_\infty^2 \left( 1 + \frac{2\gamma - 1}{3} M_\infty^2 \right) \phi_{1x}^3 \right] + \dots \end{aligned} \quad (5)$$

A comparison of coefficients of individual powers of  $\epsilon$  in equations (2) leads to the following sets of recursion formulas for the first three steps:

$$\left. \begin{aligned} \beta^2 \phi_{1x} &= \psi_{1y} \\ \phi_{1y} &= -\psi_{1x} \end{aligned} \right\} \quad (6)$$

$$\left. \begin{aligned} \beta^2 \phi_{2x} &= \psi_{2y} - \frac{1}{2} M_\infty^2 \left\{ \left[ \beta^2 + (\gamma + 1) M_\infty^2 \right] \phi_{1x}^2 + \phi_{1y}^2 + 2\phi_{1x} \phi_{1y} \right\} \\ \phi_{2y} &= -\psi_{2x} + M_\infty^2 \phi_{1x} \psi_{1x} \end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned}
 \beta^2 \phi_{3x} = \psi_{3y} - M_\infty^2 & \left[ \phi_{1x} \phi_{2x} + \phi_{1y} \phi_{2y} - \frac{1}{2} \gamma M_\infty^2 (\phi_{1x}^2 \phi_{1y}^2 - 2 \phi_{1x} \phi_{2x}) - \right. \\
 & \frac{1}{2} \gamma M_\infty^2 \left( 1 + \frac{2\gamma - 1}{3} M_\infty^2 \right) \phi_{1x}^3 - \frac{1}{2} (1 + \gamma M_\infty^2) \phi_{1x}^2 \psi_{1y} - \\
 & \left. \frac{1}{2} \phi_{1y}^2 \psi_{1y} + \phi_{2x} \psi_{1y} + \phi_{1x} \psi_{2y} \right] \\
 \phi_{3y} = -\psi_{3x} + M_\infty^2 & \left[ \phi_{1x} \psi_{2x} - \frac{1}{2} (1 + \gamma M_\infty^2) \phi_{1x}^2 \psi_{1x} - \right. \\
 & \left. \frac{1}{2} \phi_{1y}^2 \psi_{1x} + \phi_{2x} \psi_{1x} \right]
 \end{aligned} \right\} \quad (8)$$

By the introduction of the complex notation

$$x + i\beta y = z$$

$$x - i\beta y = \bar{z}$$

$$\phi_n + \frac{i}{\beta} \psi_n = w_n$$

$$\phi_n - \frac{i}{\beta} \psi_n = \bar{w}_n$$

and with the aid of the symbolic relations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = i\beta \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)$$

equations (6), (7), and (8) become, respectively,

$$w_{1\bar{z}} = 0 \quad (9)$$

$$w_{2\bar{z}} = -\frac{1}{4} M_{\infty}^2 \left[ (\sigma - 1) w_{1z}^2 + 2\sigma w_{1z} \bar{w}_{1\bar{z}} + \sigma \bar{w}_{1\bar{z}}^2 \right] \quad (10)$$

$$\begin{aligned} w_{3\bar{z}} = \frac{1}{4} M_{\infty}^2 & \left\{ 2w_{1z}(w_{2z} + \bar{w}_{2\bar{z}}) - 2\sigma(w_{1z} + \bar{w}_{1\bar{z}})(w_{2z} + w_{2\bar{z}} + \bar{w}_{2z} + \bar{w}_{2\bar{z}}) + \right. \\ & \frac{1}{4} \left[ 2 - \beta^2(4\sigma - 1) \right] w_{1z}(w_{1z}^2 - \bar{w}_{1\bar{z}}^2) + \frac{1}{8} \beta^2 (w_{1z} - \bar{w}_{1\bar{z}})^3 + \\ & \left. \left[ \frac{1}{3} + \frac{1}{8} \beta^2 - \frac{5}{6}(1 + \beta^2)\sigma + \frac{4}{3} \beta^2 \sigma^2 \right] (w_{1z} + \bar{w}_{1\bar{z}})^3 \right\} \quad (11) \end{aligned}$$

where  $\beta^2 = 1 - M_{\infty}^2$  and  $\sigma = 1 + \frac{\gamma + 1}{4} \frac{M_{\infty}^2}{\beta^2}$ . Note that the right-hand

sides of equations (10) and (11) are composed of, respectively, double and triple products of previously determined perturbation quantities. Equations (9), (10), and (11) are first-order complex-vector equations with  $w_n$  as dependent variable and  $z$  and  $\bar{z}$  as independent variables. They can be integrated in a straightforward manner. Thus, equation (9)



means that  $w_1$  is a function of  $z$  only. Then, the general integral of equation (10) is the following expression:

$$w_2 = -\frac{1}{4} M_\infty^2 \left[ (\sigma - 1) \bar{z} w_{1z}^2 + 2\sigma \bar{w}_1 w_{1z} + \sigma \int \bar{w}_{1z}^2 d\bar{z} \right] + F(z) \quad (12)$$

where  $F(z)$  is an arbitrary function of  $z$  to be determined by the boundary conditions.

From equation (12), the general integral of equation (11) is as follows:

$$\begin{aligned} w_3 = \frac{1}{4} M_\infty^2 & \left\{ \frac{1}{2} M_\infty^2 (\sigma - 1) \bar{z}^2 w_{1z}^2 w_{1zz} + \frac{1}{3} M_\infty^2 \sigma (\sigma - 1) z \bar{w}_{1z}^3 + \right. \\ & \frac{1}{2} (2\sigma - 1) \left[ \sigma (3 + \beta^2) - 2 \right] w_{1z} \int \bar{w}_{1z}^2 d\bar{z} + \\ & (\sigma - 1) \left[ \sigma (3 + \beta^2) - 1 \right] \bar{w}_1 w_{1z}^2 + \frac{1}{6} (\sigma - 1) \left[ 2\sigma (3 + \beta^2) - \right. \\ & \left. (5 + 3\beta^2) \right] \bar{z} w_{1z}^3 + \frac{1}{3} (\sigma - 1) \left[ \sigma (3 + \beta^2) - 1 \right] \int \bar{w}_{1z}^3 d\bar{z} + \\ & \frac{1}{2} \sigma (\sigma - 1) M_\infty^2 \left[ \bar{z} \bar{w}_1 \left( w_{1z}^2 \right)_z + z w_{1z} \bar{w}_{1z}^2 \right] + \frac{1}{2} \sigma^2 M_\infty^2 \left( \bar{w}_1^2 w_{1zz} + \right. \\ & \left. 2w_1 w_{1z} \bar{w}_{1z} + w_1 \bar{w}_{1z}^2 \right) \left. \right\} - \frac{1}{2} M_\infty^2 \left[ (\sigma - 1) \bar{z} w_{1z} F_z + \right. \\ & \left. \sigma \left( \bar{w}_1 F_z + w_{1z} \bar{F} \right) + \sigma \int \bar{w}_{1z} F_z d\bar{z} \right] + G(z) \quad (13) \end{aligned}$$

where  $G(z)$  is an arbitrary function of  $z$  to be determined by the boundary conditions.

It is noteworthy to remark that in the original version of the small-disturbance method the differential equations corresponding to equations (9), (10), and (11) are of the second-order Laplace and Poisson types. Therefore, the arbitrary functions added to the particular integrals can be functions of either  $z$  or  $\bar{z}$  only. Past experience, however, has shown that certain terms in the third approximation give rise to singularities in the region of flow (ref. 6). These spurious singularities must be compensated for by the addition of suitable functions of  $z$  and of  $\bar{z}$  in compliance with the boundary conditions at the surface and at infinity. In the present version of the small-disturbance method, the arbitrary functions added to the particular integrals can only be functions of  $z$ . Therefore, because both functions of  $z$  and functions of  $\bar{z}$  are necessary for the removal of external singularities in the presence of a solid boundary, the complication of apparent singularities in the external region cannot occur in the present circumstances. As a final remark, note that the form of the particular integrals obtained for subsonic flow remains the same for supersonic flow with  $\beta^2 = M_\infty^2 - 1$  and  $\sigma = 1 - \frac{\gamma - 1}{4} \frac{M_\infty^2}{\beta^2}$ .

#### SUBSONIC FLOW PAST A PARABOLIC CYLINDER

Before proceeding to the calculation of the subsonic flow, the incompressible flow past a parabolic cylinder will be derived. Thus, consider the transformation

$$z = \zeta^2$$

where  $z = x + iy$  and  $\zeta = \xi + i\eta$ . Then the real and imaginary parts give

$$\left. \begin{aligned} x &= \xi^2 - \eta^2 \\ y &= 2\xi\eta \end{aligned} \right\} \quad (14)$$

and hence, by the elimination of the variable  $\eta$ ,

$$x - \xi^2 = -\frac{y^2}{4\xi^2} \quad (15)$$

Thus, the curves for  $\xi$  constant are parabolas whose foci are at the origin (fig. 1(a)). In order to obtain the flow past a parabolic cylinder  $\xi = \xi_0$ , with undisturbed velocity from right to left, it is observed that the nondimensional stream function is given by

$$\psi = -2(\xi - \xi_0)\eta$$

The complex potential  $w = \phi + i\psi$  is then given by

$$w = -(\xi^2 - 2\xi_0\xi)$$

and the complex velocity by

$$\frac{dw}{dz} = u - iv = -1 + \frac{\xi_0}{\xi}$$

Thus,

$$\left. \begin{aligned} u &= -1 + \frac{\xi_0\xi}{\xi^2 + \eta^2} \\ v &= \frac{\xi_0\eta}{\xi^2 + \eta^2} \end{aligned} \right\} \quad (16)$$

Now, from equations (14),

$$\left. \begin{aligned} \xi^2 &= \frac{x + \sqrt{x^2 + y^2}}{2} \\ \eta^2 &= \frac{-x + \sqrt{x^2 + y^2}}{2} \end{aligned} \right\} \quad (17)$$

The polar equation of the parabola is

$$r = \frac{2\xi_0^2}{1 + \cos \theta}$$

where

$$x = r \cos \theta$$

and

$$y = r \sin \theta$$

Hence, from the second of equations (17), on the parabola  $\xi = \xi_0$ ,

$$\eta = \pm \xi_0 \tan \frac{\theta}{2}$$

Therefore, from equations (16),

$$u = -\sin^2 \frac{\theta}{2}$$

and

$$v = \pm \frac{1}{2} \sin \theta$$

or

$$q = \sqrt{u^2 + v^2} = \sin \frac{\theta}{2}$$

According to Bernoulli's theorem, the pressure coefficient  $C_p$  is

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = 1 - q^2 = \cos^2 \frac{\theta}{2}$$

Figure 2 shows the velocity and pressure-coefficient distributions along the upper surface of the parabolic cylinder. Note the monotonic character of  $q$  and that the undisturbed stream speed is reached at  $x = y = -\infty$  ( $\theta = \pi$ ). This behavior makes the parabolic cylinder a particularly good shape for the examination of the Prandtl-Busemann small-disturbance method in the neighborhood of a stagnation point.

Consider now the subsonic case; let

$$x + i\beta y = \beta^2 \left( \xi + \frac{i}{\beta} \eta \right)^2 \quad (18)$$

or

$$\left. \begin{aligned} x &= \beta^2 \left( \xi^2 - \frac{\eta^2}{\beta^2} \right) \\ y &= 2\xi\eta \end{aligned} \right\} \quad (19)$$

Elimination of  $\eta$  from these equations yields

$$x - \beta^2 \xi^2 = -\frac{y^2}{4\xi^2} \quad (20)$$

Thus, the curves for  $\xi$  constant are parabolas with foci at  $x = -M_\infty^2 \xi^2$ ,  $y = 0$  with the focal distance  $\xi^2$  independent of the stream Mach number  $M_\infty$  (fig. 1(b)). Note, further, that equation (20) results from equation (18) with the coefficient of  $i\eta$  completely arbitrary. The

choice of  $\frac{1}{\beta}$  was made in order that  $\eta$  be identical with the  $\eta$  of the incompressible case at the surface. Thus, from equations (19),

$$\left. \begin{aligned} \xi^2 &= \frac{x + \sqrt{x^2 + \beta^2 y^2}}{2\beta^2} \\ \eta^2 &= \frac{-x + \sqrt{x^2 + \beta^2 y^2}}{2} \end{aligned} \right\} \quad (21)$$

Also, on the parabola  $\xi = \xi_0$  (fig. 1(b)),

$$x = -M_\infty^2 \xi_0^2 + r \cos \theta$$

and

$$y = r \sin \theta$$

where

$$r = \frac{2\xi_0^2}{1 + \cos \theta}$$

Introducing these expressions into the second of equations (21) yields

$$\eta = \pm \xi_0 \tan \frac{\theta}{2}$$

as in the incompressible case.

Now, the first approximation  $w_1 = \phi_1 + \frac{i}{\beta} \psi_1$  is a function of  $z = x + i\beta y$  and therefore of  $\xi = \xi + \frac{i}{\beta} \eta$  only. Comparison with the incompressible case shows immediately that

$$w_1 = \phi_1 + \frac{i}{\beta} \psi_1 = 2\xi_0 \xi$$

Thus,

$$\psi = -2(\xi - \xi_0)\eta \quad (22a)$$

which satisfies the boundary condition that  $\psi = 0$  for  $\xi = \xi_0$  and that at infinity the disturbance velocities vanish for points not near the parabola. The expression for the velocity potential becomes

$$\phi = -\beta^2 \left( \xi^2 - \frac{\eta^2}{\beta^2} \right) + 2\xi_0 \xi \quad (22b)$$

Now, let the positive sense of describing the parabolic boundary be counterclockwise with the positive normal direction inward. Then, if  $ds$  and  $dn$  represent elements of arc and inward normal, respectively, the expressions for the normal velocity in terms of the stream function and the velocity potential are as follows (fig. 1(b)):

$$q_n = -\frac{\rho_\infty}{\rho} \frac{\psi_\eta}{2\sqrt{\xi_0^2 + \eta^2}} \quad (23a)$$

and

$$q_n = -\frac{1}{2(\beta^2 \xi_0^2 + \eta^2)\sqrt{\xi_0^2 + \eta^2}} \left[ (\xi_0^2 + \eta^2)\phi_\xi - M_\infty^2 \xi_0 \eta \phi_\eta \right] \quad (23b)$$

Clearly, from equations (22a) and (23a), the normal velocity vanishes along the boundary  $\xi = \xi_0$  whereas  $(q_n)_{\xi=\xi_0} \neq 0$  from equations (22b)

and (23b). Thus, in general, the boundary conditions cannot be satisfied simultaneously for both  $\phi$  and  $\psi$ . In the present version of the small-disturbance method, the function  $\phi$  does not represent the velocity potential of the flow but is utilized only for purposes of notation and ease of calculation of the stream function  $\psi$ .

Now, with  $w_1 = 2\xi_0\zeta$  and  $z = \beta^2\zeta^2$ , equation (12) yields

$$w_2 = -\frac{1}{4} \xi_0^2 \frac{M_\infty^2}{\beta^2} \left[ (\sigma - 1) \frac{\bar{\zeta}^2}{\zeta^2} + 4\sigma \frac{\bar{\zeta}}{\zeta} + 2\sigma \log \bar{\zeta} \right] + F[z(\zeta)]$$

The arbitrary function  $F$  is determined from the boundary condition that  $\psi_2 = 0$  on the parabola  $\zeta = -\bar{\zeta} + 2\xi_0$ . Thus,

$$2i\psi_2 = -\frac{1}{4} \xi_0^2 \frac{M_\infty^2}{\beta^2} \left[ (\sigma - 1) \left( \frac{\bar{\zeta}^2}{\zeta^2} - \frac{\zeta^2}{\bar{\zeta}^2} \right) + 4\sigma \left( \frac{\bar{\zeta}}{\zeta} - \frac{\zeta}{\bar{\zeta}} \right) + 2\sigma (\log \bar{\zeta} - \log \zeta) \right] + F - \bar{F}$$

where the right-hand side of this equation is a pure imaginary. Also,

$$\text{I.P.} \frac{\zeta^2}{\bar{\zeta}^2} = -\text{I.P.} \frac{\bar{\zeta}^2}{\zeta^2}$$

$$\text{I.P.} \frac{\zeta}{\bar{\zeta}} = -\text{I.P.} \frac{\bar{\zeta}}{\zeta}$$

$$\text{I.P.} \log \bar{\zeta} = -\text{I.P.} \log \zeta$$

$$\text{I.P.} \bar{F} = -\text{I.P.} F$$



Hence, on the boundary,

$$\text{I.P.F} = \frac{1}{4} \xi_0^2 \frac{M_\infty^2}{\beta^2} \text{I.P.} \left[ (\sigma - 1) \frac{(\xi - 2\xi_0)^2}{\xi^2} - 4\sigma \frac{\xi - 2\xi_0}{\xi} - 2\sigma \log \xi \right]$$

The expression on the right-hand side involves the variable  $\xi$  only and is regular throughout the field of flow; therefore,

$$F = \frac{1}{4} \xi_0^2 \frac{M_\infty^2}{\beta^2} \left[ (\sigma - 1) \frac{(\xi - 2\xi_0)^2}{\xi^2} - 4\sigma \frac{\xi - 2\xi_0}{\xi} - 2\sigma \log \xi \right]$$

and

$$w_2 = -\frac{1}{4} \xi_0^2 \frac{M_\infty^2}{\beta^2} \left[ (\sigma - 1) \frac{\bar{\xi}^2 - (\xi - 2\xi_0)^2}{\xi^2} + 4\sigma \frac{\bar{\xi} + \xi - 2\xi_0}{\xi} + 2\sigma \log \xi \bar{\xi} \right]$$

Thus,

$$\psi_2 = \text{I.P.} \cdot \beta w_2 = M_\infty^2 \xi_0^2 (\xi - \xi_0) \eta \left[ \frac{\sigma + 1}{\beta^2 \xi^2 + \eta^2} + 2(\sigma - 1) \frac{\beta^2 \xi (\xi + \xi_0)}{(\beta^2 \xi^2 + \eta^2)^2} \right] \quad (24)$$

This expression for  $\psi_2$  satisfies the boundary condition that it vanishes at the surface of the parabolic cylinder  $\xi = \xi_0$  and along the x-axis ( $y \approx \eta = 0$ ) outside the boundary. In addition, it vanishes at infinity.

The expression for the third approximation  $\psi_3$  is obtained from equation (13). In the following, there are listed some of the individual terms of equation (13) and the corresponding terms of the arbitrary function  $G[z(\xi)]$  chosen in such a way that it is regular everywhere in the field of flow and that  $\psi_3$  vanishes along the parabola  $\xi = \xi_0$  and along the x-axis  $y \approx \eta = 0$ :

$$\begin{aligned}
 \bar{z}^2 w_{1z}^2 w_{1zz} &= -\frac{\xi_0^3}{2\beta^4} \frac{\bar{\zeta}^4}{\zeta^5} & \frac{\xi_0^3}{2\beta^4} \frac{(\zeta - 2\xi_0)^4}{\zeta^5} \\
 z \bar{w}_{1\bar{z}}^3 &= \frac{\xi_0^3}{\beta^4} \frac{\bar{\zeta}^2}{\zeta^3} & \frac{\xi_0^3}{\beta^4} \frac{(\zeta - 2\xi_0)^2}{\zeta^3} \\
 w_{1z} \int \bar{w}_{1\bar{z}}^2 d\bar{z} &= \frac{2\xi_0^3}{\beta^4} \frac{1}{\zeta} \log \bar{\zeta} & - \frac{2\xi_0^3}{\beta^4} \frac{1}{\zeta - 2\xi_0} \log \frac{\zeta}{2\xi_0} \\
 \bar{w}_1 w_{1z}^2 &= \frac{2\xi_0^3}{\beta^4} \frac{\bar{\zeta}}{\zeta^2} & \frac{2\xi_0^3}{\beta^4} \frac{\zeta - 2\xi_0}{\zeta^2} \\
 \bar{z} w_{1z}^3 &= \frac{\xi_0^3}{\beta^4} \frac{\bar{\zeta}^2}{\zeta^3} & - \frac{\xi_0^3}{\beta^4} \frac{(\zeta - 2\xi_0)^2}{\zeta^3} \\
 \int \bar{w}_{1\bar{z}}^3 d\bar{z} &= -\frac{2\xi_0^3}{\beta^4} \frac{1}{\zeta} & - \frac{2\xi_0^3}{\beta^4} \frac{1}{\zeta} \\
 \bar{z} \bar{w}_1 (w_{1z}^2)_z &= -\frac{2\xi_0^3}{\beta^4} \frac{\bar{\zeta}^3}{\zeta^4} & - \frac{2\xi_0^3}{\beta^4} \frac{(\zeta - 2\xi_0)^3}{\zeta^4} \\
 z w_{1z} \bar{w}_{1\bar{z}}^2 &= \frac{\xi_0^3}{\beta^4} \frac{\zeta}{\zeta^2} & - \frac{\xi_0^3}{\beta^4} \frac{\zeta - 2\xi_0}{\zeta^2} \\
 \bar{w}_1^2 w_{1zz} &= -\frac{2\xi_0^3}{\beta^4} \frac{\bar{\zeta}^2}{\zeta^3} & \frac{2\xi_0^3}{\beta^4} \frac{(\zeta - 2\xi_0)^2}{\zeta^3} \\
 2w_1 w_{1z} \bar{w}_{1\bar{z}} &= \frac{4\xi_0^3}{\beta^4} \frac{1}{\zeta} & \frac{4\xi_0^3}{\beta^4} \frac{1}{\zeta} \\
 w_1 \bar{w}_{1\bar{z}}^2 &= \frac{2\xi_0^3}{\beta^4} \frac{\zeta}{\zeta^2} & - \frac{2\xi_0^3}{\beta^4} \frac{\zeta - 2\xi_0}{\zeta^2}
 \end{aligned} \tag{25}$$

Thus,

$$\begin{aligned}
 G = & \frac{1}{4} \xi_0^3 \frac{M_\infty^2}{\beta^4} \left( \frac{1}{4} M_\infty^2 (\sigma - 1)^2 \frac{(\xi - 2\xi_0)^4}{\xi^5} + \left\{ \frac{1}{3} M_\infty^2 \sigma (4\sigma - 1) - \right. \right. \\
 & \left. \frac{1}{6} (\sigma - 1) [2\sigma(3 + \beta^2) - (5 + 3\beta^2)] \right\} \frac{(\xi - 2\xi_0)^2}{\xi^3} - \\
 & (2\sigma - 1) [\sigma(3 + \beta^2) - 2] \frac{1}{\xi - 2\xi_0} \log \frac{\xi}{2\xi_0} + \left\{ -\frac{1}{2} M_\infty^2 \sigma (3\sigma - 1) + \right. \\
 & \left. 2(\sigma - 1) [\sigma(3 + \beta^2) - 1] \right\} \frac{\xi - 2\xi_0}{\xi^2} + 2 \left\{ M_\infty^2 \sigma^2 - \right. \\
 & \left. \frac{1}{3} (\sigma - 1) [\sigma(3 + \beta^2) - 1] \right\} \frac{1}{\xi} - M_\infty^2 \sigma (\sigma - 1) \frac{(\xi - 2\xi_0)^3}{\xi^4} \Bigg) - \\
 & \frac{1}{8} \xi_0^3 \frac{M_\infty^4}{\beta^4} \left\{ (\sigma - 1) \left[ 4(\sigma - 1) \xi_0^2 \frac{(\xi - 2\xi_0)^2}{\xi^5} + 2(\sigma + 1) \xi_0 \frac{(\xi - 2\xi_0)^2}{\xi^4} + \right. \right. \\
 & \left. \left. \sigma \frac{(\xi - 2\xi_0)^2}{\xi^3} \right] - 2\sigma \left[ 4(\sigma - 1) \xi_0^2 \frac{\xi - 2\xi_0}{\xi^4} + 2(\sigma + 1) \xi_0 \frac{\xi - 2\xi_0}{\xi^3} + \right. \right. \\
 & \left. \left. \sigma \frac{\xi - 2\xi_0}{\xi^2} \right] + \sigma \left[ -(\sigma - 1) \frac{\xi - 2\xi_0}{\xi^2} + 4\sigma \frac{1}{\xi} + \right. \right. \\
 & \left. \left. 2\sigma \frac{1}{\xi - 2\xi_0} \log \frac{\xi}{2\xi_0} \right] + 2\sigma \left[ \frac{4}{3} (\sigma - 1) \frac{\xi_0^2}{\xi^3} + (\sigma + 1) \frac{\xi_0}{\xi^2} + \frac{\sigma}{\xi} \right] \right\}
 \end{aligned}$$

This expression for  $G$  is regular throughout the field of flow, since the apparent pole at  $\xi = 2\xi_0$  is canceled by the zero of  $\log \frac{\xi}{2\xi_0}$  there.

Then the expression for  $w_3$  is given by

$$\begin{aligned}
 w_3 = & \frac{1}{4} \xi_0^3 \frac{M_\infty^2}{\beta^4} \left( \frac{1}{4} M_\infty^2 (\sigma - 1)^2 \frac{-\xi^4 + (\xi - 2\xi_0)^4}{\xi^5} - \sigma(\sigma - 1) M_\infty^2 \frac{\xi^3 + (\xi - 2\xi_0)^3}{\xi^4} + \right. \\
 & \frac{1}{3} M_\infty^2 \sigma(\sigma - 1) \frac{\xi^5 + \xi^3(\xi - 2\xi_0)^2}{\xi^3 \xi^3} + \left\{ \frac{1}{6} (\sigma - 1) [2\sigma(3 + \beta^2) - (5 + 3\beta^2)] - \right. \\
 & \left. \sigma^2 M_\infty^2 \right\} \frac{\xi^2 - (\xi - 2\xi_0)^2}{\xi^3} + 4(\sigma - 1) [\sigma(3 + \beta^2) - 1] \frac{\xi - \xi_0}{\xi^2} + \\
 & \frac{1}{2} \sigma(3\sigma - 1) M_\infty^2 \frac{\xi^3 - \xi^2(\xi - 2\xi_0)}{\xi^2 \xi^2} - \frac{1}{3} \left\{ (\sigma - 1) [\sigma(3 + \beta^2) - 1] - 3\sigma^2 M_\infty^2 \right\} \frac{\xi}{\xi^5} + \\
 & (2\sigma - 1) [\sigma(3 + \beta^2) - 2] \left( \frac{1}{\xi} \log \frac{\xi}{2\xi_0} - \frac{1}{\xi - 2\xi_0} \log \frac{\xi}{2\xi_0} \right) + \frac{1}{8} \xi_0^3 \frac{M_\infty^4}{\beta^4} \left\{ (\sigma - 1) \left[ 4(\sigma - 1) \frac{\xi_0^2}{\xi^5} + \right. \right. \\
 & \left. \left. 2(\sigma + 1) \frac{\xi_0}{\xi^4} + \frac{\sigma}{\xi^3} \right] \left[ \xi^2 - (\xi - 2\xi_0)^2 \right] + 2\sigma \left[ 4(\sigma - 1) \frac{\xi_0^2}{\xi^4} + 2(\sigma + 1) \frac{\xi_0}{\xi^3} + \right. \right. \\
 & \left. \left. \frac{\sigma}{\xi^2} \right] \left[ \xi + (\xi - 2\xi_0) \right] - \sigma \left[ (\sigma - 1) \frac{\xi(\xi - 2\xi_0)^2 - \xi^2(\xi - 2\xi_0)}{\xi^2 \xi^2} - 4\sigma \frac{(\xi - 2\xi_0) - \xi}{\xi^2} - \right. \right. \\
 & \left. \left. \frac{2\sigma}{\xi} \log \frac{\xi}{2\xi_0} + \frac{2\sigma}{\xi - 2\xi_0} \log \frac{\xi}{2\xi_0} \right] - 2\sigma \left[ \frac{4}{3} (\sigma - 1) \xi_0^2 \left( \frac{1}{\xi^3} + \frac{1}{\xi^3} \right) + (\sigma + 1) \xi_0 \left( \frac{1}{\xi^2} + \frac{1}{\xi^2} \right) + \sigma \left( \frac{1}{\xi} + \frac{1}{\xi} \right) \right] \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 \psi_3 = & \frac{1}{2} \xi_0^3 M_\infty^4 (\sigma - 1)^2 \frac{(\xi - \xi_0)\eta}{(\beta^2 \xi^2 + \eta^2)^5} \left[ \beta^6 \xi^4 (\xi^3 + 5\xi_0 \xi^2 - 2\xi_0^2 \xi - 10\xi_0^3) - \right. \\
 & \beta^4 \eta^2 \xi^2 (11\xi^3 + 15\xi_0 \xi^2 - 20\xi_0^2 \xi - 20\xi_0^3) + \beta^2 \eta^4 (15\xi^3 + 11\xi_0 \xi^2 - 10\xi_0^2 \xi - 2\xi_0^3) - \\
 & \left. \eta^6 (5\xi + \xi_0) \right] + \xi_0^3 M_\infty^4 \sigma (\sigma - 1) \frac{(\xi - \xi_0)\eta}{(\beta^2 \xi^2 + \eta^2)^4} \left[ \beta^4 \xi^3 (2\xi^2 - 3\xi_0 \xi - 8\xi_0^2) - \right. \\
 & 2\beta^2 \eta^2 \xi (4\xi^2 + \xi_0 \xi - 4\xi_0^2) + \eta^4 (6\xi + \xi_0) \left. \right] + \xi_0^3 M_\infty^2 \frac{(\xi - \xi_0)\eta}{(\beta^2 \xi^2 + \eta^2)^3} \left\{ \sigma^2 M_\infty^2 - \right. \\
 & \left. \frac{1}{6} (\sigma - 1) [\sigma(7 + \beta^2) - (5 + 3\beta^2)] \right\} \left[ \beta^2 \xi^2 (\xi + 3\xi_0) - (3\xi + \xi_0)\eta^2 \right] - \\
 & \xi_0^4 M_\infty^4 \sigma (\sigma + 1) \frac{(\xi - \xi_0)\eta}{(\beta^2 \xi^2 + \eta^2)^3} (3\beta^2 \xi^2 - \eta^2) - \frac{1}{2} \xi_0^3 M_\infty^2 (\sigma - 1) \left\{ [\sigma(11 + 5\beta^2) - 4] \xi - \right. \\
 & \left. \sigma M_\infty^2 \xi_0 \right\} \frac{(\xi - \xi_0)\eta}{(\beta^2 \xi^2 + \eta^2)^2} + \frac{1}{2} \xi_0^3 M_\infty^2 [\sigma(\sigma - 1)(7 + \beta^2) + 2] \frac{\xi - \xi_0}{(\beta^2 \xi^2 + \eta^2) [\beta^2 (\xi - 2\xi_0)^2 + \eta^2]} \left[ \eta \log \frac{\beta^2 \xi^2 + \eta^2}{4\beta^2 \xi_0^2} - \right. \\
 & \left. \frac{\beta^2 (\xi^2 - 2\xi_0 \xi) + \eta^2}{\beta \xi_0} \tan^{-1} \frac{\eta}{\beta \xi} \right] \quad (26)
 \end{aligned}$$

This expression for  $\psi_3$  satisfies the boundary condition that it vanishes at the surface of the parabolic cylinder  $\xi = \xi_0$  and along the x-axis ( $y \approx \eta = 0$ ) outside the boundary. The complete expression for the stream function  $\psi$ , inclusive of the third approximation, is then given by the sum of equations (22a), (24), and (26).

For the purpose of comparison with the Janzen-Rayleigh solution (to be derived in the following section), the expression for the fluid speed at the boundary will now be obtained in the form of a power-series development in  $M_\infty^2$ . Symbolically,

$$\frac{\partial}{\partial x} = \frac{1}{2(\beta^2 \xi^2 + \eta^2)} \left( \xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right)$$

and

$$\frac{\partial}{\partial y} = \frac{1}{2(\beta^2 \xi^2 + \eta^2)} \left( \eta \frac{\partial}{\partial \xi} + \beta^2 \xi \frac{\partial}{\partial \eta} \right)$$

Also, in terms of the relevant quantities,

$$\frac{\rho_\infty}{\rho} = 1 - \frac{M_\infty^2}{\beta^2} \psi_{1y} + \frac{M_\infty^2}{\beta^2} \left[ \frac{1}{2} (\psi_{1x}^2 + \psi_{1y}^2) - \psi_{2y} + \psi_{1x} \psi_{2x} + \right. \\ \left. \psi_{1y} \psi_{2y} - \psi_{3y} \right] + \dots$$

Then, if only the terms that involve the Mach number up to  $M_\infty^2$  are retained,

$$\psi = -2(\xi - \xi_0)\eta + 2M_\infty^2 \xi_0^2 (\xi - \xi_0) \frac{\eta}{\xi^2 + \eta^2} + M_\infty^2 \left[ \xi_0 \eta \log \frac{\xi^2 + \eta^2}{4\xi_0^2} - \right. \\ \left. (\xi^2 + \eta^2 - 2\xi_0\xi) \tan^{-1} \frac{\eta}{\xi} \right] \frac{\xi_0^3 (\xi - \xi_0)}{(\xi^2 + \eta^2) (\xi^2 + \eta^2 - 4\xi_0\xi + 4\xi_0^2)}$$

and

$$\frac{\rho_\infty}{\rho} = 1 - \frac{M_\infty^2}{\xi^2 + \eta^2} \left( \xi_0 \xi - \frac{1}{2} \xi_0^2 \right)$$

Now, along the surface of the cylinder  $\xi = \xi_0$ , where most of the interest centers,

$$q = - \frac{\rho_\infty \sqrt{\xi_0^2 + \eta^2}}{2\rho (\beta^2 \xi_0^2 + \eta^2)} (\psi_\xi)_{\xi=\xi_0}$$

Hence, with

$$\frac{\rho_\infty}{\rho} = 1 - \frac{1}{2} M_\infty^2 \frac{\xi_o^2}{\xi_o^2 + \eta^2}$$

and

$$(\psi_\xi)_{\xi=\xi_o} = -2\eta + 2M_\infty^2 \frac{\xi_o^2 \eta}{\xi_o^2 + \eta^2} + M_\infty^2 \left[ \xi_o \eta \log \frac{\xi_o^2 + \eta^2}{4\xi_o^2} + \right. \\ \left. (\xi_o^2 - \eta^2) \tan^{-1} \frac{\eta}{\xi_o} \right] \frac{\xi_o^3}{(\xi_o^2 + \eta^2)^2}$$

it follows that

$$q = \frac{1}{2\sqrt{\xi_o^2 + \eta^2}} \left\{ 2\eta - M_\infty^2 \frac{\xi_o^2 \eta}{\xi_o^2 + \eta^2} - M_\infty^2 \frac{\xi_o^3}{(\xi_o^2 + \eta^2)^2} \left[ \xi_o \eta \log \frac{\xi_o^2 + \eta^2}{4\xi_o^2} + \right. \right. \\ \left. \left. (\xi_o^2 - \eta^2) \tan^{-1} \frac{\eta}{\xi_o} \right] \right\} \quad (27)$$

#### JANZEN-RAYLEIGH METHOD FOR PARABOLIC CYLINDER

The solution of the problem of subsonic flow past a parabolic cylinder by means of the Janzen-Rayleigh method follows along the same lines as that by means of the small-disturbance method. Thus,

$$\left. \begin{aligned}
 \phi &= \phi_0 + M_\infty^2 \phi_1 + \dots \\
 \text{and} \\
 \psi &= \psi_0 + M_\infty^2 \psi_1 + \dots
 \end{aligned} \right\} \quad (28)$$

where  $\phi_0$  and  $\psi_0$  are, respectively, the incompressible velocity potential and stream function. Again, from equation (3), expanding in powers of  $M_\infty^2$  yields

$$\frac{\rho_\infty}{\rho} = 1 - \frac{1}{2} M_\infty^2 \left[ 1 - (\phi_{ox}^2 + \phi_{oy}^2) \right] + \dots \quad (29)$$

Then, from equations (2), a comparison of the various powers of  $M_\infty^2$  yields the following recursive system of first-order equations for  $\phi_0, \psi_0; \phi_1, \psi_1; \dots$ :

$$\left. \begin{aligned}
 \phi_{ox} &= \psi_{oy} \\
 \phi_{oy} &= -\psi_{ox}
 \end{aligned} \right\} \quad (30)$$

$$\left. \begin{aligned}
 \phi_{1x} &= \psi_{1y} - \frac{1}{2} \psi_{oy} \left[ 1 - (\phi_{ox}^2 + \phi_{oy}^2) \right] \\
 \phi_{1y} &= -\psi_{1x} + \frac{1}{2} \psi_{ox} \left[ 1 - (\phi_{ox}^2 + \phi_{oy}^2) \right]
 \end{aligned} \right\} \quad (31)$$

Now, let

$$\begin{aligned}
 x + iy &= z \\
 x - iy &= \bar{z} \\
 \phi_n + i\psi_n &= w_n \\
 \phi_n - i\psi_n &= \bar{w}_n
 \end{aligned}$$



Equations (30) and (31) then become

$$w_{0\bar{z}} = 0 \quad (32)$$

and

$$w_{1\bar{z}} = -\frac{1}{4} \bar{w}_{0\bar{z}} + \frac{1}{4} w_{0z} \bar{w}_{0\bar{z}}^2 \quad (33)$$

Equation (32) means that  $w_0$  is a function of  $z$  only, and the general integral of equation (33) is

$$w_1 = -\frac{1}{4} w_0 + \frac{1}{4} w_{0z} \int \bar{w}_{0\bar{z}}^2 d\bar{z} + H(z) \quad (34)$$

where  $H(z)$  is an arbitrary function to be determined according to the boundary conditions.

For the parabolic cylinder,

$$z = \zeta^2 \quad (\zeta = \xi + i\eta)$$

and

$$w_0 = -(\zeta - \xi_0)^2$$

From equation (34) then

$$w_1 = \frac{1}{2} \xi_0 \bar{\zeta} - \frac{1}{2} \xi_0^2 \log \bar{\zeta} + \frac{1}{4} \xi_0 \frac{\bar{\zeta}^2}{\zeta} - \xi_0^2 \frac{\bar{\zeta}}{\zeta} + \frac{1}{2} \frac{\xi_0^3}{\zeta} \log \bar{\zeta} + H[z(\zeta)]$$

The arbitrary function  $H$  is determined from the boundary condition that  $\psi_1 = 0$  on the parabola  $\bar{\zeta} = -\zeta + 2\xi_0$ . Thus,

$$2i\psi_1 = \frac{1}{2} \xi_0 (\bar{\zeta} - \zeta) - \frac{1}{2} \xi_0^2 (\log \bar{\zeta} - \log \zeta) + \frac{1}{4} \xi_0 \left( \frac{\bar{\zeta}^2}{\zeta} - \frac{\zeta^2}{\bar{\zeta}} \right) - \xi_0^2 \left( \frac{\bar{\zeta}}{\zeta} - \frac{\zeta}{\bar{\zeta}} \right) + \frac{1}{2} \xi_0^3 \left( \frac{1}{\zeta} \log \bar{\zeta} - \frac{1}{\bar{\zeta}} \log \zeta \right) + H - \bar{H}$$

where the right-hand side of this equation is a pure imaginary. Also,  $\text{I.P.}f(\zeta) = -\text{I.P.}\bar{f}(\bar{\zeta})$ ; hence, on the boundary,

$$\text{I.P.}H = \text{I.P.} \left[ \frac{1}{2} \xi_0 \zeta - \frac{1}{2} \xi_0^2 \log \frac{\zeta}{2\xi_0} - \frac{1}{4} \xi_0 \frac{(\zeta - 2\xi_0)^2}{\zeta} - \xi_0^2 \frac{\zeta - 2\xi_0}{\zeta} - \frac{1}{2} \frac{\xi_0^3}{\zeta - 2\xi_0} \log \frac{\zeta}{2\xi_0} \right]$$

The expression on the right-hand side involves the variable  $\zeta$  only and is regular throughout the field of flow; therefore,

$$H = \frac{1}{2} \xi_0 \zeta - \frac{1}{2} \xi_0^2 \log \frac{\zeta}{2\xi_0} - \frac{1}{4} \xi_0 \frac{(\zeta - 2\xi_0)^2}{\zeta} - \xi_0^2 \frac{\zeta - 2\xi_0}{\zeta} - \frac{1}{2} \frac{\xi_0^3}{\zeta - 2\xi_0} \log \frac{\zeta}{2\xi_0}$$

and

$$w_1 = \frac{1}{2} \xi_0 (\xi + \bar{\xi}) - \frac{1}{2} \xi_0^2 \log \frac{\xi \bar{\xi}}{4 \xi_0^2} + \frac{1}{4} \xi_0 \frac{\bar{\xi}^2 - (\xi - 2\xi_0)^2}{\xi} -$$

$$\xi_0^2 \frac{\bar{\xi} + \xi - 2\xi_0}{\xi} + \frac{1}{2} \xi_0^3 \left( \frac{1}{\xi} \log \frac{\bar{\xi}}{2\xi_0} - \frac{1}{\xi - 2\xi_0} \log \frac{\xi}{2\xi_0} \right) \quad (35)$$

Then,

$$\phi_1 = \xi_0 \xi - \frac{1}{2} \xi_0^2 \log \frac{\xi^2 + \eta^2}{4 \xi_0^2} - \frac{\xi_0 (\xi - \xi_0) (\xi_0 \xi + \eta^2)}{\xi^2 + \eta^2} -$$

$$\frac{1}{2} \left[ \xi_0 (\xi^2 - 2\xi_0 \xi - \eta^2) \log \frac{\xi^2 + \eta^2}{4 \xi_0^2} + 2\eta (\xi^2 + \eta^2 - 2\xi_0 \xi + \right.$$

$$\left. 2\xi_0^2) \tan^{-1} \frac{\eta}{\xi} \right] \frac{\xi_0^3}{(\xi^2 + \eta^2) (\xi^2 + \eta^2 - 4\xi_0 \xi + 4\xi_0^2)} \quad (36)$$

and

$$\psi_1 = - \frac{\xi_0 (\xi - \xi_0)^2 \eta}{\xi^2 + \eta^2} + \left[ \xi_0 \eta \log \frac{\xi^2 + \eta^2}{4 \xi_0^2} - (\xi^2 + \eta^2 - \right.$$

$$\left. 2\xi_0 \xi) \tan^{-1} \frac{\eta}{\xi} \right] \frac{\xi_0^3 (\xi - \xi_0)}{(\xi^2 + \eta^2) (\xi^2 + \eta^2 - 4\xi_0 \xi + 4\xi_0^2)} \quad (37)$$

Usually, most of the interest lies in the velocity distribution on the boundary. Thus, substitute  $\xi = \xi_0$  into equation (36) and in the -R.P.  $(\xi - \xi_0)^2$ ; then at the boundary

$$\phi = \eta^2 - \frac{1}{2} M_\infty^2 \frac{\xi_0^2 \eta}{\xi_0^2 + \eta^2} \left( \eta \log \frac{\xi_0^2 + \eta^2}{4\xi_0^2} + 2\xi_0 \tan^{-1} \frac{\eta}{\xi_0} \right)$$

The velocity of the fluid is given by

$$q = \frac{1}{2\sqrt{\xi_0^2 + \eta^2}} (\phi_\eta)_{\xi=\xi_0}$$

Hence,

$$q = \frac{1}{2\sqrt{\xi_0^2 + \eta^2}} \left\{ 2\eta - M_\infty^2 \frac{\xi_0^2 \eta}{\xi_0^2 + \eta^2} - M_\infty^2 \frac{\xi_0^3}{(\xi_0^2 + \eta^2)^2} \left[ \xi_0 \eta \log \frac{\xi_0^2 + \eta^2}{4\xi_0^2} + (\xi_0^2 - \eta^2) \tan^{-1} \frac{\eta}{\xi_0} \right] \right\}$$

This equation is in agreement with equation (27). Thus, the small-disturbance solution when expanded in powers of  $M_\infty^2$  yields precisely the Janzen-Rayleigh result. Now, at the upper surface,  $\eta = \xi_0 \tan \frac{\theta}{2}$ ; therefore,

$$q = \sin \frac{\theta}{2} - \frac{1}{4} M_\infty^2 \cos^2 \frac{\theta}{2} \left\{ 2 \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \left[ \theta \cos \theta - 2 \sin \theta \log \left( 2 \cos \frac{\theta}{2} \right) \right] \right\} \quad (38)$$

and

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho_\infty U^2}$$

$$= \cos^2 \frac{\theta}{2} \left( 1 + \frac{1}{4} M_\infty^2 \left\{ 1 + 3 \sin^2 \frac{\theta}{2} + \sin \theta \left[ \theta \cos \theta - 2 \sin \theta \log \left( 2 \cos \frac{\theta}{2} \right) \right] \right\} \right) \quad (39)$$

Table I lists values of  $q$  and  $C_p$  for  $M_\infty = 0.5$  over the upper surface of the cylinder and figure 2 shows the corresponding graphs.

Since the completion of this paper the attention of the author has been drawn to a recent calculation by Imai of the Janzen-Rayleigh solution for a parabola including terms in  $M_\infty^4$  (ref. 7). An error in sign in reference 7 has been corrected. (Note last term in equation (38)).

#### DISCUSSION OF ANALYSIS

The main concern of this paper has been the presentation of Imai's elegant version of the small-disturbance method and its application to the problem of two-dimensional compressible flow past a parabolic cylinder. The example of the parabolic cylinder was chosen for the dual purpose of illustrating the results of the general analysis and for comparison with the  $M_\infty^2$ -expansion or Janzen-Rayleigh method of solution.

One of the basic assumptions of the small-disturbance method is that, except for a small region in the neighborhood of the nose, the derivatives of  $\psi_n$  must be less than those of  $\psi_m$  where  $m < n$ . A numerical comparison at the surface of the parabolic cylinder shows, however, that  $\psi_{3x} > \psi_{2x}$  over a large portion of the surface. Presumably, if it were possible to calculate  $\psi_n$  to any order, the condition that  $\psi_{nx} < \psi_{mx}$  for  $n > m$  would be satisfied from some definite value of  $m$  onward. Although the solution as obtained in the present paper satisfies the boundary conditions at the surface and at infinity for each step of the iteration process, violation of the foregoing basic assumption renders

the curtailed solution useless for the calculation of the velocity and pressure distributions at the surface. Nevertheless, some interesting results with regard to thin-airfoil theory can be obtained from the expressions for  $\psi_1$ ,  $\psi_2$ , and  $\psi_3$  obtained from equations (22a), (24), and (26). Thus, by following the ideas of Van Dyke (ref. 8), the velocity at the surface of the parabolic cylinder is developed in a series of powers of  $\frac{2\xi_0^2}{\eta^2}$ , the ratio of the radius of curvature at the vertex and the abscissa measured from the vertex. For this purpose, consider the following expression for the fluid speed  $q$  in terms of the derivatives  $\psi_n$ :

$$\begin{aligned}
 q = 1 + \frac{1}{\beta^2} & \left( -\psi_{1y} - \psi_{2y} - \psi_{3y} + \frac{1}{2} \psi_{1x}^2 + \frac{1}{2} \psi_{1y}^2 + \frac{1}{2} \psi_{2y}^2 + \right. \\
 & \left. \psi_{1x}\psi_{2x} + \psi_{1y}\psi_{2y} \right) + \frac{1}{8\beta^4} \left( 4\alpha M_\infty^2 - 1 \right) \left( \psi_{1x}^4 + 8\psi_{1y}\psi_{2y} + \right. \\
 & \left. 4\psi_{1y}^2 + 4\psi_{2y}^2 - 4\psi_{1x}^2\psi_{1y} - 4\psi_{1x}^2\psi_{2y} \right) \quad (40)
 \end{aligned}$$

where only those terms have been retained which involve the ratio  $\frac{2\xi_0^2}{\eta^2}$  inclusive of the second power. From equations (22a), (24), and (26), the required expressions for the derivatives of  $\psi_1$ ,  $\psi_2$ , and  $\psi_3$  at the surface of the parabolic cylinder are as follows:

$$\left. \begin{aligned}
 \psi_{1x} &= -\frac{\xi_0}{\eta} \left( 1 - \beta^2 \frac{\xi_0^2}{\eta^2} \right) \\
 \psi_{2x} &= \frac{1}{2} M_\infty^2 (\sigma + 1) \frac{\xi_0^3}{\eta^3} \\
 \psi_{1y} &= \beta^2 \frac{\xi_0^2}{\eta^2} \left( 1 - \beta^2 \frac{\xi_0^2}{\eta^2} \right) \\
 \psi_{2y} &= \frac{1}{2} M_\infty^2 \left[ (\sigma + 1) \frac{\xi_0^2}{\eta^2} + 2(\sigma - 3) \beta^2 \frac{\xi_0^4}{\eta^4} \right] \\
 \psi_{3y} &= M_\infty^2 \frac{\xi_0^4}{\eta^4} \left\{ \frac{1}{3} \sigma^2 (1 - 5\beta^2) - \frac{\sigma}{6} (3 - \beta^2) + \frac{5}{6} (1 - 3\beta^2) - \right. \\
 &\quad \left. \frac{1}{4} \left[ \sigma(\sigma - 1)(7 + \beta^2) + 2 \right] \left( 2 \log 2\beta \frac{\xi_0}{\eta} + \frac{1}{\beta \frac{\xi_0}{\eta}} \tan^{-1} \frac{1}{\beta \frac{\xi_0}{\eta}} \right) \right\}
 \end{aligned} \right\} \quad (41)$$

Substitution of these expressions into equation (40) then yields

$$\begin{aligned}
 q = 1 - \frac{\xi_0^2}{\eta^2} \left( 1 + \sigma \frac{M_\infty^2}{\beta^2} \right) + \frac{\xi_0^4}{2\eta^4} & \left\{ 1 + \frac{1}{4\beta^4} (4\sigma M_\infty^2 - 1) \left[ 1 + (\sigma - 1) M_\infty^2 (1 + \beta^2 + \sigma M_\infty^2) \right] + \right. \\
 \frac{1}{4} \frac{M_\infty^4}{\beta^2} (\sigma + 1)^2 - \frac{1}{3} \frac{M_\infty^2}{\beta^2} & \left[ 2\sigma^2 (1 - 5\beta^2) + 4\sigma\beta^2 + 8 - 27\beta^2 \right] + \\
 \left. \frac{1}{2} \frac{M_\infty^2}{\beta^2} \left[ \sigma(\sigma - 1)(7 + \beta^2) + 2 \right] \right. & \left. \left( 2 \log 2\beta \frac{\xi_0}{\eta} + \frac{1}{\beta \frac{\xi_0}{\eta}} \tan^{-1} \frac{1}{\beta \frac{\xi_0}{\eta}} \right) \right\} \quad (42)
 \end{aligned}$$

The first two terms on the right-hand side of this equation agree with the results of second-order thin-airfoil theory (ref. 8). The third term presumably would be obtained from a third-order thin-airfoil theory. Although this term has been written in a form to suggest that it is of

order  $\left( \frac{2\xi_0^2}{\eta^2} \right)^2$ , it is of interest to note that the expression

$\frac{1}{\beta \frac{\xi_0}{\eta}} \tan^{-1} \frac{1}{\beta \frac{\xi_0}{\eta}}$  can be expanded as follows:

$$\frac{1}{\beta \frac{\xi_0}{\eta}} \tan^{-1} \frac{1}{\beta \frac{\xi_0}{\eta}} = \frac{\pi}{2\beta \frac{\xi_0}{\eta}} - 1 + \frac{1}{3} \left( \beta \frac{\xi_0}{\eta} \right)^2 - \dots \quad \left( \beta \frac{\xi_0}{\eta} \leq 1 \right)$$

This expansion leads to a term of the order  $\frac{3}{2}$ ; namely,

$$\frac{\sqrt{2} \pi}{32} \frac{M_\infty^2}{\beta^3} \left[ \sigma(\sigma - 1)(7 + \beta^2) + 2 \right] \left( \frac{2\xi_0^2}{\eta^2} \right)^{3/2} \quad (43)$$



The third-order thin-airfoil result can then be written as follows:

$$q = 1 - \frac{1}{4} \left( \frac{2\xi_o^2}{\beta^2 \eta^2} \right) \left[ 1 + (\sigma - 1) M_\infty^2 \right] + \frac{\sqrt{2\pi}}{32} M_\infty^2 \left[ \sigma(\sigma - 1)(7 + \beta^2) + 2 \right] \left( \frac{2\xi_o^2}{\beta^2 \eta^2} \right)^{3/2} \quad (44)$$

The small circles in figure 2 designate points calculated by means of this equation with  $M_\infty = 0.5$ . The first circle corresponds to  $\theta = 120^\circ$

for which the parameter  $\frac{2\xi_o^2}{\eta^2}$  equals  $\frac{2}{3}$ . If the terms that involve

$\left( \frac{2\xi_o^2}{\eta^2} \right)^2$  are included, the magnitude of  $q$  becomes greater than unity

even for values of  $\theta$  corresponding to points relatively far from the nose. For example,  $q = 1.0132$  at the point corresponding to  $\theta = 120^\circ$ . These spurious values of  $q$  indicate that the fourth approximation of the small-disturbance method must contribute additional terms of the

order  $\left( \frac{2\xi_o^2}{\eta^2} \right)^2$ .

Langley Aeronautical Laboratory,  
National Advisory Committee for Aeronautics,  
Langley Field, Va., September 21, 1954.

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TABLE I  
VELOCITY AND PRESSURE-COEFFICIENT DISTRIBUTIONS

$\theta$ , deg	$M_\infty = 0$		$M_\infty = 0.50$	
	$q$	$C_p$	$q$	$C_p$
0	0	1.0000	0	1.0625
5	.0436	.9981	.0403	1.0607
15	.1305	.9830	.1207	1.0459
30	.2588	.9330	.2402	.9971
45	.3827	.8536	.3573	.9185
60	.5000	.7500	.4705	.8147
75	.6088	.6294	.5781	.6915
90	.7071	.5000	.6782	.5565
105	.7934	.3706	.7687	.4184
120	.8660	.2500	.8472	.2866
135	.9239	.1465	.9115	.1707
150	.9659	.0670	.9596	.0795
165	.9914	.0170	.9896	.0207
180	1.0000	0	1.0000	0



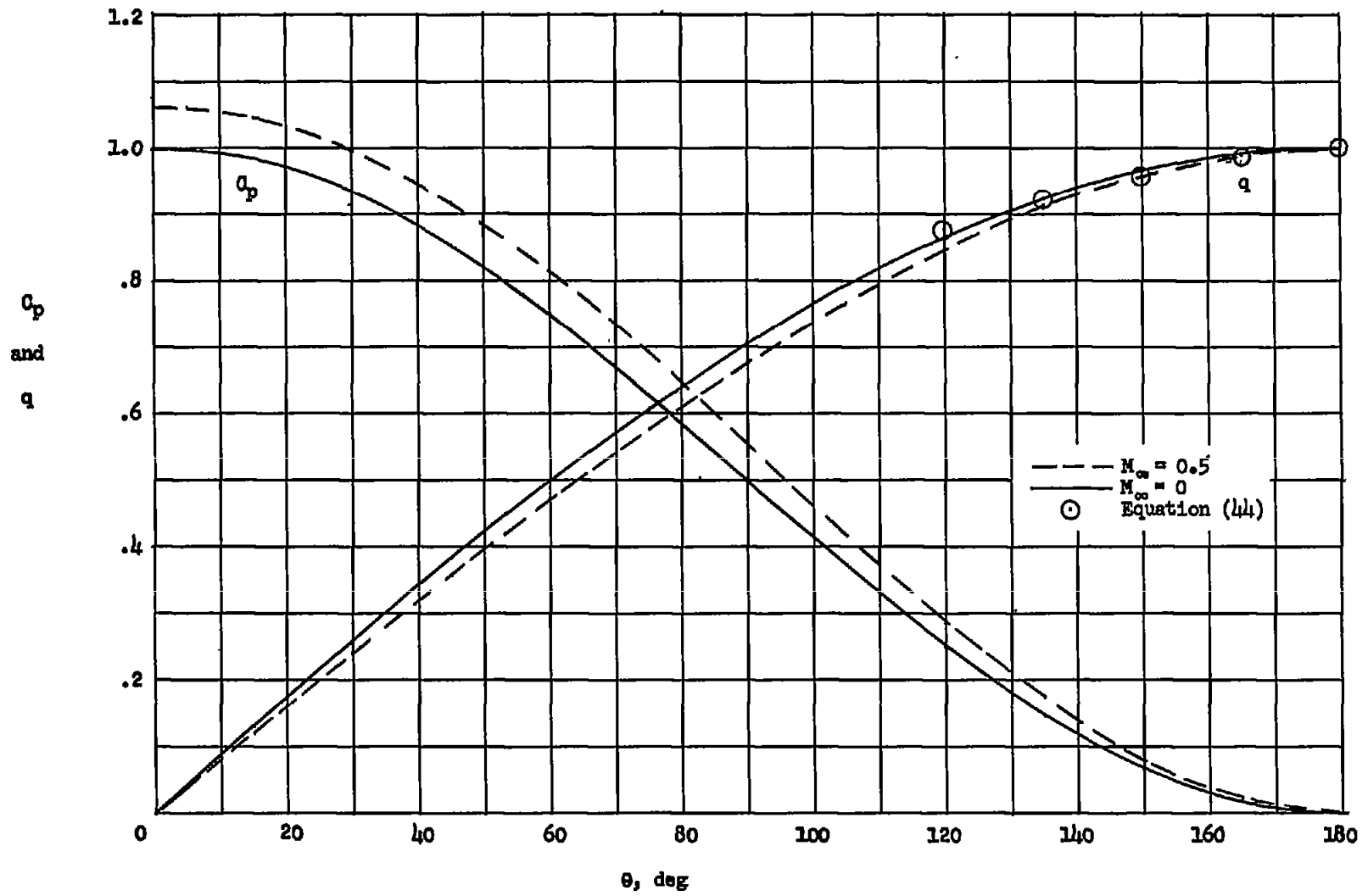


Figure 2.- Velocity and pressure-coefficient distribution along upper surface of parabolic cylinder.